JOURNAL OF APPROXIMATION THEORY 34, 115–130 (1982)

Existence of Pointwise-Lipschitz-Continuous Selections of the Metric Projection for a Class of Z-Spaces

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Communicated by E. W. Cheney

Received December 18, 1980

We study the problem of existence of pointwise-Lipschitz-continuous selections for the metric projection. We first approximate by finite dimensional subspaces of $C(X)$ where X is a certain compact Hausdorff space and give a sufficient condition for existence of such selections. We apply this result to the case when X is the union of finitely many compact real intervals and get in this case a partial converse to a recent result of Brown.

INTRODUCTION

Let X be a compact Hausdorff space and $C(X)$ the space of all real-valued continuous functions f on X under the uniform norm $\|f\|:=\sup\{|f(x)|: x\in X\}$. If G is a subspace of $C(X)$, then for all $f\in C(X)$ the set $P_G(f):=\{g_0\in G: \|f-g_0\|=\inf\{\|f-g\|: g\in G\}\}$ is the *set* of *best uniform approximations* to f from G. This defines a set-valued mapping P_G which is called the *metric projection* onto G. A mapping $s: C(X)\to G$ is called a *selection* for P_G if $s(f)\in P_G(f)$ for all $f\in C(X)$. Furthermore, a selection s for P_G is called *pointwise-Lipschitz-continuous* if, for each $f\in C(X)$, there exists a constant $K_f>0$ such that, for each $\tilde{f}\in C(X)$, $\|s(f)-s(\tilde{f})\|\leqslant K_f\|f-\tilde{f}\|$ (this clearly implies that s is continuous). A selection s is called *quasilinear* if, for each $f\in C(X)$, for each $g\in G$ and for all constants c, d, the relation $s(cf+dg)=cs(f)+dg$ holds.

In this paper we study the problem of existence of continuous and pointwise-Lipschitz-continuous, quasilinear selections for metric projections. This problem has been considered by a number of authors in the last years. Lazar *et al.* [8] have been the first to study this problem. They have characterized those one-dimensional subspaces G of $C(X)$ which admit continuous selections for P_G. By using the theory of weak Chebyshev spaces,

0021-9045/82/020115-16\$02.00/0

Nürnberger and Sommer (see [11, 13, 14, 16–19]) have completely characterized those finite dimensional subspaces G of $C[a, b]$, where $[a, b]$ is a compact real interval, which admit continuous selections for P_G. Recently, Blatt *et al.* [1] have shown that each continuous selection constructed by Nürnberger and Sommer is even pointwise-Lipschitz-continuous which is a strong property of the metric projection and also quasilinear. For locally compact subspaces X of the real line Nürnberger [12] has been able to show the existence of continuous selections for P_G, in case G is an element of a class of finite dimensional weak Chebyshev subspaces of $C(X)$ and Deutsch and Kenderov [5] have studied this problem in the case when X is a normed linear space.

Recently, Brown [4] has been concerned with finite dimensional subspaces G of $C(X)$ with the property that no non-zero function in G vanishes on all points of some non-empty open subset of X. Such spaces are called *Z-spaces*. The preceding property implies that if G is a Z-space with dimension at least two, then X can have no isolated points. Brown has given a description of those X for which there is a Z-subspace G admitting a continuous selection for P_G. For example, one of his main results which is an extension of Mairhuber's theorem is the following: Suppose that there exists a Z-subspace G of $C(X)$ with dimension at least two such that there is a continuous selection for P_G. If X is metrisable, then X is homeomorphic to a subspace of a circle.

Using the arguments established by Brown it is easily verified that the existence of a continuous selection for P_G, where G is an n-dimensional subspace of $C(X)$, implies that each non-zero $g \in G$ has at most n distinct zeros on X and at most $n-1$ zeros with a sign change in X (Lemma 1.1). In this paper we study the problem of conversing this statement. We are therefore concerned with those n-dimensional subspaces G of $C(X)$ whose non-zero elements have only finitely many zeros. In [20] we have shown that, under appropriate hypothesis on X, there is a class of these spaces such that for each G contained in this class each $f \in C(X)$ has a particular best approximation $g_0 \in P_G(f)$ which is called alternation element (Definition 1.4, Theorem 1.5). In that paper we furthermore have given a sufficient condition for uniqueness of alternation elements (Theorem 1.5). This result immediately applies to our studies because the property that each $f \in C(X)$ has a unique alternation element $g_f \in P_G(f)$ implies the existence of a quasilinear, pointwise-Lipschitz-continuous selection s defined by $s(f) := g_f$ (Theorem 1.7).

In Section 2 we apply our results established in Section 1 to the case when $X = \bigcup_{j=1}^{l} I_j$, the union of finitely many compact real intervals. We show the following result which gives for this particular X a partial converse to the results of Brown: Let G be an n-dimensional subspace of $C(X)$. If each $g \in G$ has at most $n-1$ zeros with sign changes and if there is a $z \in X$ such that G

satisfies the Haar condition on $X\backslash\{z\}$, then there exists a pointwise-Lipschitz-continuous, quasilinear selection for P_G (Theorem 2.1). This theorem yields the first result on existence of spaces G with dimension at least two admitting continuous and even pointwise-Lipschitz-continuous selections for P_G although G is *not* Chebyshev and also *not* weak Chebyshev (Example 2). Therefore this situation is quite different from the case $X=[a,b]$, because there the weak Chebyshev property is necessary for existence of continuous selections (see Nürnberger [11]). Our results and also, for one-dimensional spaces, the results of Lazar *et al.* [8] show that the number of the *zeros with a sign change* of the functions in G plays the fundamental role for existence or nonexistence of continuous selections, however, *not* the number of the sign changes while for $X=[a,b]$ the continuity of the functions in G does not allow a difference between sign changes and zeros with a sign change.

Using the arguments established in this paper it is easily verified that all results given here are also true if X will be replaced by a corresponding locally compact Hausdorff space T and $C(X)$ by $C_0(T)$, the space of all real-valued continuous functions f on T vanishing at infinity, i.e., for each $\varepsilon>0$ the set $\{x\in T: |f(x)|\geqslant\varepsilon\}$ is compact.

1. A Sufficient Condition for the Existence of Pointwise-Lipschitz-Continuous Selections

In the following X will be any compact Hausdorff space and $\hat{X}$ a compact Hausdorff space satisfying the following property: For each sequence $\{x_k\}\subset\hat{X}$ converging to $x\in\hat{X}$ and each neighborhood U of x there is an integer k_0 such that for all points $x_k, x_{\tilde{k}}\in U$, $k\geqslant k_0$, $\tilde{k}\geqslant k_0$, there is a path P from x_k to $x_{\tilde{k}}$ completely contained in U.

Furthermore, in the following G will always denote an n-dimensional subspace of $C(X)$ and of $C(\hat{X})$, resp. with $n\geqslant 2$ and X resp. $\hat{X}$ will contain at least one non-isolated point. For brevity we will give some notations and definitions only for the more general space X but we always will assume that the same has been done for the space $\hat{X}$.

Recently, Brown [4] has given a description of those X for which there is a Z-subspace G with dimension at least two admitting a continuous selection for P_G. Following his arguments in the proofs of Lemma 3 and Lemma 6 the following statement is easily verified.

LEMMA 1.1. *If G is a Z-subspace of $C(X)$ with dimension n such that G admits a continuous selection for P_G, then each non-zero $g\in G$ has at most n distinct zeros on X and at most $n-1$ zeros with a sign change in X.*

We say that a function $f \in C(X)$ has a *zero* x *with a sign change* in X if for each neighborhood U of x there are two points $y_1, y_2 \in U$ such that $f(y_1) f(y_2) < 0$.

If $X = [a, b]$, a real compact interval, then the converse to Lemma 1.1 follows directly from the results of Nürnberger and Sommer [13]. Therefore we conjecture that the converse also holds in our general situation.

In the following we will study this problem. We first introduce an important class of n-dimensional subspaces of $C(X)$.

DEFINITION 1.2. We say that G satisfies the *Haar condition* on a subset Y of X if each non-zero $g \in G$ has at most $n-1$ zeros on Y. G is said to be *Chebyshev* if $P_G(f)$ is a singleton for each $f \in C(X)$.

The proof of the following classical result can be found in the book of Meinardus [9].

THEOREM 1.3. *The following statements are equivalent:*

(i) *G is Chebyshev.*

(ii) *G satisfies the Haar condition on X.*

(iii)
$$\det(g_i(x_j))_{i,j=1}^n := \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \vdots & & \vdots \\ g_n(x_1) & \cdots & g_n(x_n) \end{vmatrix} \neq 0$$

for each basis $g_1, ..., g_n$ of G and all n distinct points $x_1, ..., x_n \in X$.

For brevity we set $D_G(x_1, ..., x_n) := \det(g_i(x_j))_{i,j=1}^n$ for all points $x_1, ..., x_n \in X$, where $g_1, ..., g_n$ is a fixed chosen basis of G.

Henceforth we will suppose that G satisfies the following conditions:

(1.1) There is a *minimal* finite subset $Z = \{z_1, ..., z_m\}$ of *non-isolated* points of X such that G satisfies the Haar condition on $X \backslash Z$.

(1.2) For any n distinct points $x_1, ..., x_n \in X$ there are pairwise disjoint neighborhoods U_i of x_i, $i = 1, ..., n$, such that $\varepsilon D_G(y_1, ..., y_n) \geqslant 0$, $\varepsilon = \pm 1$ for all $y_i \in U_i$, $i = 1, ..., n$.

Then these both conditions imply that for any n distinct points $x_1, ..., x_n \in X$ the inequality $\varepsilon D_G(y_1, ..., y_n) > 0$, $\varepsilon = \pm 1$, holds for all those n-tuples $(y_1, ..., y_n) \in \prod_{i=1}^n U_i$ for which $\{y_1, ..., y_n\} \cap Z = \varnothing$. This will play an important role for the existence of particular best approximations.

The preceding arguments allow the following notation: Let any $n+1$ distinct points $x_0, ..., x_n \in X$ be given. Then for each subset $\{x_0, ..., x_{i-1},$

$x_{i+1},\dots, x_n\}$ of $\{x_0,\dots, x_n\}$ there are neighborhoods U_j of x_j, $j=0,\dots, n$, $j\neq i$ such that $\varepsilon D_G(y_0,\dots, y_{i-1}, y_{i+1},\dots, y_n)>0$, $\varepsilon=\pm 1$, for all those n-tuples

$$(y_0,\dots, y_{i-1}, y_{i+1},\dots, y_n)\in \prod_{\substack{j=0\\ j\neq i}}^{n} U_j$$

for which $\{y_0,\dots, y_{i-1}, y_{i+1},\dots, y_n\}\cap Z=\varnothing$. We set:

$$\Delta_i(x_0,\dots, x_n):=\operatorname{sgn} D_G(y_0,\dots, y_{i-1}, y_{i+1},\dots, y_n).$$

Using this notation we define certain best approximations as follows.

DEFINITION 1.4. If $f\in C(X)$, then $g_0\in P_G(f)$ is said to be an *alternation element* (*AE*) of f, if there exist $n+1$ distinct points $x_0,\dots, x_n\in X$ such that

$$\varepsilon(-1)^i\Delta_i(x_0,\dots, x_n)(f-g_0)(x_i)=\|f-g_0\|, \qquad i=0,\dots, n,\ \varepsilon=\pm 1.$$

The points $x_0,\dots, x_n$ are called *oriented extreme points* (*OE-points*) of $f-g_0$.

In [20] we have shown the following results on existence and uniqueness of AEs which are the key results for existence of pointwise-Lipschitz-continuous selections.

THEOREM 1.5. *Let G be an n-dimensional subspace of $C(X)$ and of $C(\hat{X})$, respectively such that G satisfies conditions* (1.1) *and* (1.2). *Then the following statements hold:*

(i) *Each $f\in C(\hat{X})$ has at least one* AE $g_0\in P_G(f)$.

(ii) *If Z is a singleton, then $G\subset C(X)$ implies that each $f\in C(X)$ has at most one* AE *and $G\subset C(\hat{X})$ implies that each $f\in C(\hat{X})$ has a unique* AE.

(iii) *If each $f\in C(X)$ has a unique* AE, *then each non-zero $g\in G$ has at most n distinct zeros.*

We will now show that uniqueness of an AE always implies the existence of a pointwise-Lipschitz-continuous selection. To prove this we first give the following lemma:

LEMMA 1.6. *Let each $g\in G$ have at most n distinct zeros and let $n+1$ distinct points $t_0,\dots, t_n\in X$ be given. If $Z\cap\{t_0,\dots, t_n\}\neq\varnothing$, then there is an integer $i\in\{0,\dots, n\}$ such that $t_i\in Z$ and $D_G(t_0,\dots, t_{i-1}, t_{i+1},\dots, t_n)\neq 0$.*

Proof. Let $n+1$ distinct points $t_0,\dots, t_n\in X$ be given such that $\{t_0,\dots, t_n\}\cap Z\neq\varnothing$. Then without loss of generality we may assume that $Z\cap\{t_0,\dots, t_n\}=\{t_0,\dots, t_m\}$. Now suppose that for each $i=0,\dots, m$, $D_G(t_0,\dots, t_{i-1}, t_{i+1},\dots, t_n)=0$. This implies the existence of non-zero functions $g_i \in G, i=0,\dots,m$, satisfying $g_i(t_j)=0$ for $j=0,\dots,n, j \neq i$. Since each non-zero $g \in G$ has at most n distinct zeros, it follows that $g_i(t_i) \neq 0$. Therefore the functions $g_0,\dots,g_m$ are linearly independent on X. Furthermore it follows from $\{t_{m+1},\dots,t_n\} \cap Z = \varnothing$ and condition (1.1) that for each $i=m+1,\dots,n$ there is a linearly independent function $g_i \in G$ satisfying $g_i(t_i)=1$ and $g_i(t_j)=0$ for $j=m+1,\dots,n, j \neq i$. Summarizing all these arguments we have got $n+1$ linearly independent functions in G which contradicts the hypothesis that $\dim G = n$.

We are now in a position to prove the main result of this section.

THEOREM 1.7. *If for each $f \in C(\hat{X})$ there exists a unique* AE $g_f \in P_G(f)$, *then the map $s: C(\hat{X}) \to G$ defined by $s(f) := g_f$ is a quasilinear, pointwise-Lipschitz-continuous selection for P_G.*

Proof. We first show that s is continuous at f. Suppose that s is not continuous at f. Then there exist functions $f \in C(\hat{X})$, $g_0 \in P_G(f)$ and a sequence $\{f_k\} \subset C(\hat{X})$ such that $f_k \to f$, $s(f_k) \to g_0$ but $g_0 \neq s(f) = g_f$. We will show that g_0 is also an AE of f and this will contradict the uniqueness of g_f.

Since, for each k, $s(f_k)$ is an AE of f_k, for each k there exist $n+1$ OE-points $t_{0k},\dots,t_{nk}$ such that

$$\begin{aligned}\varepsilon_k(-1)^i \Delta_i(t_{0k},\dots,t_{nk})(f_k - s(f_k))(t_{ik})\\ = \|f_k - s(f_k)\|, \qquad i=0,\dots,n,\ \varepsilon_k = \pm 1.\end{aligned}$$

Without loss of generality we may assume that $\varepsilon_k = \varepsilon$ and $t_{ik} \to t_i \in \hat{X}$. Then following the proof of Theorem 1.5 in [20] we can show that all points t_i are distinct and $\Delta_i(t_{0k},\dots,t_{nk}) \to \Delta_i(t_0,\dots,t_n)$ for $k \to \infty$ (in [20] we only have been able to prove this for the case that G is an n-dimensional subspace of $C(\hat{X})$. Therefore we have chosen here the same hypothesis on G. All following arguments in this proof are even true if we replace $\hat{X}$ by X).

Therefore g_0 must be an AE of f and thus we have shown that s is continuous at f.

Using the proof of Corollary 1.3 in [12] it is easily verified that s is quasilinear.

We now prove that s is even pointwise-Lipschitz-continuous. Suppose that s is not pointwise-Lipschitz-continuous. Then there exist a function $f \in C(\hat{X})$ and a sequence $\{f_k\} \subset C(\hat{X})$ such that for each k

$$\|s(f_k) - s(f)\| > k\,\|f_k - f\|. \tag{*}$$

We may assume that $f_k \to f$, since otherwise there exist a subsequence of $\{f_k\}$

which we again denote by $\{f_k\}$ and a constant $K_0 > 0$ such that for each k, $\|f-f_k\| > K_0$. This implies that for each k

$$\begin{aligned}\|s(f)-s(f_k)\| &\leqslant \|f_k - s(f_k)\| + \|f-s(f)\| + \|f-f_k\| \\ &\leqslant \|f_k - s(f)\| + \|f-s(f)\| + \|f-f_k\| \\ &\leqslant 2(\|f-s(f)\| + \|f-f_k\|) \\ &\leqslant 2(\|f\| + \|f-f_k\|) \\ &\leqslant 2\left(\frac{\|f\|}{K_0}+1\right)\|f-f_k\|\end{aligned}$$

which is a contradiction to inequality (*).

Since for each k, $s(f_k)$ is an AE of f_k, for each k there exist $n+1$ OE-points $t_{0k},..., t_{nk} \in \hat{X}$ such that

$$\begin{aligned}&\varepsilon_k(-1)^i \Delta_i(t_{0k},..., t_{nk})(f_k - s(f_k))(t_{ik}) \\ &\quad = \|f_k - s(f_k)\|, \qquad i=0,..., n,\ \varepsilon_k = \pm 1.\end{aligned}$$

We may assume that, for each k, $\varepsilon_k = \varepsilon$. Since s is continuous at f and $f_k \to f$, we furthermore may assume that there are $n+1$ distinct points $t_0,..., t_n \in \hat{X}$ such that $t_{ik} \to t_i$ and

$$\varepsilon(-1)^i \Delta_i(t_0,..., t_n)(f-s(f))(t_i) = \|f-s(f)\|, \qquad i=0,..., n.$$

We will now show the existence of a constant $C>0$ such that $\|s(f)-s(f_k)\| \leqslant C\|f-f_k\|$ which will contradict inequality (*). To do this we will study the behavior of the functions $s(f)-s(f_k)$ on a certain subset U_j of $\hat{X}$. For defining this set we first select a point $\bar{t}_j \in \{t_0,..., t_n\}$ as follows:

If $Z \cap \{t_0,..., t_n\} = \varnothing$, then condition (1.1) implies that for $j=0,..., n$, $D_G(t_0,..., t_{j-1}, t_{j+1},..., t_n) \neq 0$. If there is a non-isolated point $t_j \in \{t_0,..., t_n\}$, then we set $\bar{t}_j := t_j$. If not, then we choose an arbitrary point $z \in Z$, which is non-isolated by definition of Z, and set $\bar{t}_0 := z$.

If $\{t_0,..., t_n\} \cap Z \neq 0$, then using Theorem 1.5(iii) and Lemma 1.6 we obtain a point $t_j \in Z \cap \{t_0,..., t_n\}$ such that $D_G(t_0,..., t_{j-1}, t_{j+1},..., t_n) \neq 0$. We set $\bar{t}_j := t_j$.

This point $\bar{t}_j$ leads to a subset U_j of $\hat{X}$ as follows: We first define $n+1$ points $\hat{t}_0,..., \hat{t}_n$ by $\hat{t}_i := t_i$ for $i=0,..., n, i \neq j$ and $\hat{t}_j := \bar{t}_j$. Then for $i=0,..., n$ condition (1.2) of G implies the existence of closed neighborhoods U_{il}, $l=0,..., n$, $l \neq i$ of $\hat{t}_l$ such that $\delta_i D_G(\tilde{t}_0,..., \tilde{t}_{i-1}, \tilde{t}_{i+1},..., \tilde{t}_n) > 0$, $\delta_i = \pm 1$, for all n-tuples

$$(\tilde{t}_0,..., \tilde{t}_{i-1}, \tilde{t}_{i+1},..., \tilde{t}_n) \in \prod_{\substack{l=0\\ l\neq i}}^{n} U_{il}$$

for which

$$\{\tilde{t}_0,\dots,\tilde{t}_{i-1},\tilde{t}_{i+1},\dots,\tilde{t}_n\}\cap Z=\varnothing.$$

We set

$$U_l:=\bigcap_{\substack{i=0\\ i\neq l}}^{n} U_{il}.$$

Each U_l is a closed neighborhood of $\hat{t}_l$. This implies the existence of an integer k_0 such that $t_{lk}\in U_l$ for $l=0,\dots,n$, $l\neq j$, and all $k\geqslant k_0$. Therefore it follows that

$$\Delta_i(t_{0k},\dots,t_{j-1,k},x,t_{j+1,k},\dots,t_{nk})=\Delta_i(t_0,\dots,t_{j-1},x,t_{j+1},\dots,t_n)$$

for all $x\in U_j$ and all $k\geqslant k_0$.

If $\bar{t}_j\neq t_j$, then following the construction of $\bar{t}_j$ we see that $t_0,\dots,t_n$ are isolated points. In this case there must even be an integer $k_1\geqslant k_0$ such that $t_{ik}=t_i$ for $i=0,\dots,n$ and all $k\geqslant k_1$. For estimating the functions $s(f)-s(f_k)$ we need a special basis of G. It follows from $D_G(t_0,\dots,t_{j-1},t_{j+1},\dots,t_n)\neq 0$ that for $i=0,\dots,n$, $i\neq j$ there are functions $g_i\in G$ defined by

$$g_i(x):=c_iD_G(t_0,\dots,t_{i-1},x,t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_n)$$

satisfying $g_i(t_i)\neq 0$, where $c_i:=\Delta_i(t_0,\dots,t_{j-1},t_j,t_{j+1},\dots,t_n)/\Delta_i(t_0,\dots,t_{j-1},\bar{t}_j,t_{j+1},\dots,t_n)$. Then it is easily verified that $g_0,\dots,g_{j-1},g_{j+1},\dots,g_n$ form a basis of G (the constants c_i are only essential in the case when $\bar{t}_j\neq t_j$, i.e., in the case when all points $t_0,\dots,t_n$ are isolated). Therefore each $g\in G$ can be written as

$$g=\sum_{\substack{i=0\\ i\neq j}}^{n} g(t_i)\,g_i.$$

In particular, for each k we have

$$\begin{aligned}
\varepsilon(-1)^{j+1}(s(f)-s(f_k))(x)&=\sum_{\substack{i=0\\ i\neq j}}^{n}\varepsilon(-1)^{j+1}(s(f)-s(f_k))(t_i)\cdot g_i(x)\\
&=\sum_{\substack{i=0\\ i\neq j}}^{n}\varepsilon(-1)^{j+1}(s(f)-s(f_k))(t_i)\cdot\operatorname{sgn} g_i(x)\cdot|g_i(x)|\\
&=\sum_{\substack{i=0\\ i\neq j}}^{n}\varepsilon(-1)^{j+1}(s(f)-s(f_k))(t_i)\cdot c_i(-1)^{i+j+1}\\
&\quad\times\operatorname{sgn} D_G(t_0,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},x,t_{j+1},\dots,t_n)\,|g_i(x)|
\end{aligned}$$

$$= \sum_{\substack{i=0\\ i\neq j}}^{n} \varepsilon(-1)^i (s(f)-s(f_k))(t_i)\, c_i$$
$$\times \operatorname{sgn} D_G(t_0,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},x,t_{j+1},\dots,t_n)\,|\,g_i(x)|.$$

We furthermore need the inequality

$$\begin{aligned}
&\varepsilon(-1)^i \Delta_i(t_0,\dots,t_n)(s(f)-s(f_k))(t_i)\\
&\quad = \varepsilon(-1)^i \Delta_i(t_0,\dots,t_n)(s(f)-f+f_k-s(f_k)+f-f_k)(t_i)\\
&\quad \leqslant -\|f-s(f)\| + \|f_k-s(f_k)\| + \|f-f_k\|\\
&\quad \leqslant -\|f-s(f)\| + \|f_k-s(f)\| + \|f-f_k\|\\
&\quad \leqslant -\|f-s(f)\| + \|f_k-f\| + \|f-s(f)\| + \|f-f_k\|\\
&\quad = 2\,\|f-f_k\|.
\end{aligned}$$

Analogously we can show that for each k,

$$\varepsilon(-1)^i \Delta_i(t_{0k},\dots,t_{nk})(s(f_k)-s(f))(t_{ik}) \leqslant 2\,\|f-f_k\|.$$

Summarizing all preceding arguments, for all k and all $x \in U_j$ we obtain the relation

$$\begin{aligned}
&\varepsilon(-1)^{j+1}(s(f)-s(f_k))(x)\\
&\quad = \sum_{\substack{i=0\\ i\neq j}}^{n} \varepsilon(-1)^i (s(f)-s(f_k))(t_i)\, c_i\\
&\qquad \times \operatorname{sgn} D_G(t_0,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},x,t_{j+1},\dots,t_n)\,|\,g_i(x)|\\
&\quad = \sum_{i\in I} \varepsilon(-1)^i (s(f)-s(f_k))(t_i)\, c_i\\
&\qquad \times \operatorname{sgn} D_G(t_0,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},x,t_{j+1},\dots,t_n)\,|\,g_i(x)|\\
&\qquad (\text{where } I:=\{i\in\{0,\dots,n\}, i\neq j:\\
&\qquad \operatorname{sgn} D_G(t_0,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},x,t_{j+1},\dots,t_n)\neq 0\})\\
&\quad = \sum_{i\in I} \varepsilon(-1)^i (s(f)-s(f_k))(t_i)\, \Delta_i(t_0,\dots,t_n)|\,g_i(x)|\\
&\quad \leqslant \sum_{i\in I} 2\|f-f_k\|\,|\,g_i(x)|\\
&\quad \leqslant 2\,\|f-f_k\| \sum_{\substack{i=0\\ i\neq j}}^{n} \|\,g_i\| = M_1\,\|f-f_k\|,
\end{aligned}$$

where

$$M_1 := 2 \sum_{\substack{i=0 \\ i \neq j}}^{n} \| g_i \|.$$

Now choosing for each k the same integer j as for $t_0, \ldots, t_n$ we can easily show that for each $x \in U_j$ and each $k \geqslant k_1$ the inequality

$$\varepsilon(-1)^{j+1}(s(f) - s(f_k))(x) \geqslant -2 \|f - f_k\| \sum_{\substack{i=0 \\ i \neq j}}^{n} \| g_{ik} \| = -M_2 \|f - f_k\|$$

holds, where

$$M_2 := 2 \sup_{k \geqslant k_1} \sum_{\substack{i=0 \\ i \neq j}}^{n} \| g_{ik} \|$$

and for $i = 0, \ldots, n$, $i \neq j$, g_{ik} is defined by

$$g_{ik}(x) := c_{ik} D_G(t_{0k}, \ldots, t_{i-1,k}, x, t_{i+1,k}, \ldots, t_{j-1,k}, t_{j+1,k}, \ldots, t_{nk})$$

with

$$c_{ik} := \frac{\Delta_i(t_{0k}, \ldots, t_{j-1,k}, t_{jk}, t_{j+1,k}, \ldots, t_{nk})}{\Delta_i(t_{0k}, \ldots, t_{j-1,k}, \bar{t}_j, t_{j+1,k}, \ldots, t_{nk})}.$$

Since $g_{ik} \to g_i$ for $k \to \infty$, it follows the existence of a positive constant $M \geqslant \max\{M_1, M_2\}$ such that for all $k \geqslant k_1$

$$\|s(f) - s(f_k)\|_{U_j} \leqslant M \|f - f_k\|.$$

By construction the point $\bar{t}_j$ is non-isolated. This guarantees that the neighborhood U_j of $\bar{t}_j$ contains infinitely many elements. Then, since by Theorem 1.5(iii) each non-zero $g \in G$ has at most n distinct zeros, compactness arguments imply that there is a positive constant L satisfying $\min_{\|g\|=1} \|g\|_{U_j} = L$. Setting $g := (s(f) - s(f_k)) \| s(f) - s(f_k)\|$ we obtain the inequality

$$\|s(f) - s(f_k)\| \leqslant \frac{1}{L} \|s(f) - s(f_k)\|_{U_j} \leqslant \frac{M}{L} \|f - f_k\|$$

for all $k \geqslant k_1$ which contradicts inequality (*).

This shows that s is pointwise-Lipschitz-continuous and completes the proof.

2. Applications and Examples

This section is concerned with linear subspaces G of $C(\hat{X})$, where $\hat{X}$ is chosen as

$$\hat{X} = \bigcup_{j=1}^{l} I_j,$$

the union of finitely many *non-degenerate* real compact intervals.

In the following let G again denote an n-dimensional subspace of $C(\hat{X})$, $n \geqslant 2$. Using the results proved in Section 1 we will give a partial converse to Lemma 1.1 by showing that for all elements of a class of n-dimensional Z-subspaces of $C(\hat{X})$ uniqueness of the AEs is satisfied. Then Theorem 1.7 will yield the desired result.

The main result of this section can now be stated as follows.

THEOREM 2.1. *Let each $g \in G$ have at most $n-1$ zeros with sign changes in $\hat{X}$ and let $z \in \hat{X}$ such that G satisfies the Haar condition on $\hat{X} \setminus \{z\}$. Then there exists a unique pointwise-Lipschitz-continuous quasilinear selection for P_G.*

The uniqueness of such a selection follows directly from a result of Garkavi [7]. This author has shown that for any finite dimensional Z-subspace G of $C(\hat{X})$ the set of functions $f \in C(\hat{X})$ having a unique best approximation with respect to G lies dense in $C(\hat{X})$. Therefore we may expect at most one continuous selection.

To prove the existence we have only to show that G satisfies condition (1.2) while condition (1.1) is trivially satisfied. Then the statements of Theorem 1.5(ii) and of Theorem 1.7 complete the proof. Note that each $\hat{X}$ defined in this section obviously has the additional properties as have been required in Section 1. Furthermore one can see that the particular choice of $\hat{X}$ ensures that each G satisfying the hypothesis of Theorem 2.1 must be a *Z-subspace* of $C(\hat{X})$. Therefore Theorem 2.1 yields a partial converse to Lemma 1.1.

By the preceding arguments the next lemma will complete the proof of Theorem 2.1. To prove this we will need some properties of weak Chebyshev spaces as have been shown in [21]. An n-dimensional subspace G of $C_0(T)$, where T is any locally compact subset of the real line is said to be *weak Chebyshev* if each $g \in G$ has at most $n-1$ sign changes, i.e., there do not exist points $t_0 < t_1 < \cdots < t_n$ in T with $g(t_{i-1})\, g(t_i) < 0$ for $i = 1,\dots, n$.

LEMMA 2.2. *Let each non-zero $g \in G$ have at most n distinct zeros on $\hat{X}$ and at most $n-1$ zeros with a sign change in $\hat{X}$. Then the following statements hold:*

(i) *There exists a minimal finite set of points* $Z = \{z_1, ..., z_m\}$ *such that* G *satisfies the Haar condition on* $\hat{X}\backslash Z$.

(*ii*) *For any* n *distinct points* $x_1, ..., x_n \in \hat{X}$ *there are neighborhoods* $U_i, i = 1, ..., n$ *of* x_i *such that* $\varepsilon D_G(y_1, ..., y_n) > 0$, $\varepsilon = \pm 1$, *for all n-tuples* $(y_1, ..., y_n) \in \prod_{i=1}^n U_i$ *for which* $\{y_1, ..., y_n\} \cap Z = \varnothing$.

Proof. (*i*) Since no non-zero $g \in G$ vanishes on an open subinterval of $\hat{X}$ and each $g \in G$ has at most $n-1$ zeros with a sign change, for every $j \in \{1, ..., l\}$, $G|_{I_j}$ is a weak Chebyshev subspace with dimension n. Furthermore, by hypothesis, each non-zero $g \in G$ has at most n distinct zeros on $\hat{X}$. Therefore if, for some $j \in \{1, ..., l\}$, G does not satisfy the Haar condition on I_j, using Theorem 4.6 in Sommer and Strauss [21], we can conclude that there is a point $y_j \in I_j$ such that G satisfies the Haar condition on $I_j\backslash\{y_j\}$ and, in case $y_j \in \operatorname{int} I_j$, $g(y_j) = 0$ for all $g \in G$. We distinguish:

If for some $j \in \{1, ..., l\}$, $y_j \in \operatorname{int} I_j$, then by the preceding argument no non-zero $g \in G$ can have n distinct zeros on $\hat{X}\backslash\{y_j\}$ and, setting $Z = \{y_j\}$, the statement is proved. But if, for all $j \in \{1, ..., l\}$, G satisfies the Haar condition on $\operatorname{int} I_j$, then, setting $\tilde{Z} = bd\,\hat{X}$, the statement will follow for $\tilde{Z}$ instead of Z. Then this implies that there is a minimal subset Z of $\tilde{Z}$ with the desired property. We first should observe that the particular choice of $\hat{X}$ implies that $\tilde{Z}$ is a finite set. Now suppose that G does not satisfy the Haar condition on $\hat{X}\backslash bd\,\hat{X}$. Then there must be n distinct points $x_1 < \cdots < x_n$ in $\operatorname{int} \hat{X}$ and a non-zero function $g_0 \in G$ with $g_0(x_i) = 0$ for $i = 1, ..., n$. Since g_0 may not have n zeros with a sign change, we may assume that there is a positive constant ε such that $g_0(x) > 0$ on $[x_1 - \varepsilon, x_1 + \varepsilon]$, $x \neq x_1$, where $[x_1 - \varepsilon, x_1 + \varepsilon] \subset I_j \subset \hat{X}$. Now choosing a point $x_0 \in [x_1 - \varepsilon, x_1)$ and following the proof of Lemma 1.6 it turns out that there is an $i \in \{0, ..., n\}$ such that $D_G(x_0, ..., x_{i-1}, x_{i+1}, ..., x_n) \neq 0$. Then $D_G(x_1, ..., x_n) = 0$ implies that $i \geqslant 1$. We distinguish:

First case. All zeros $x_2, ..., x_{i-1}, x_{i+1}, ..., x_n$ are zeros with a sign change. Since by hypothesis G satisfies the Haar condition on $\operatorname{int} I_j$ and $x_1 \in \operatorname{int} I_j$, there is a $\tilde{g} \in G$ with $\tilde{g}(x_1) > 0$. Then for a sufficiently small positive constant c the function $g_0 - c\tilde{g}$ has at least n zeros with a sign change in $\hat{X}$ which contradicts the hypotheses on G.

Second case. There is a further zero x_l, $l \geqslant 2$, $l \neq i$, of g_0 such that g_0 does not change the sign at x_l. Then $D_G(x_0, ..., x_{i-1}, x_{i+1}, ..., x_n) \neq 0$ implies the existence of a function $h_0 \in G$ satisfying $h_0(x_0) = 1$ and $h_0(x_j) = \operatorname{sgn} g_0(x_j - \delta_j)$, $j = 1, ..., n, j \neq i$, with $\delta_j > 0$ sufficiently small. Then it is easily verified that for a sufficiently small constant c the function $g_0 - ch_0$ has at least n zeros with a sign change in $\hat{X}$ which is a contradiction again.

Thus we have shown that G satisfies the Haar condition on $\hat{X}\backslash bd\hat{X}$.

(ii) Let Z be a minimal finite subset of $\hat{X}$ such that G satisfies the Haar condition on $\hat{X}\backslash Z$. The existence of such a set Z follows from statement (i). Let any n distinct points $x_1,\ldots,x_n \in \hat{X}$ be given. If $\{x_1,\ldots,x_n\} \cap Z = \emptyset$, then statement (i) implies that $D_G(x_1,\ldots,x_n) \neq 0$. By the continuity of D_G at the n-tuple $(x_1,\ldots,x_n)$ the statement is proved.

Therefore we must study the case when $\{x_1,\ldots,x_n\} \cap Z \neq \emptyset$. We distinguish three cases.

First case. $Z = \{z\}$ and $z \in \operatorname{int} \hat{X}$. Then from the arguments established in (i) it follows that $g(z) = 0$ for all $g \in G$. This means that for all points $x_2,\ldots,x_n \in \hat{X}$, $D_G(z, x_2,\ldots,x_n) = 0$. Now suppose that for some distinct points $x_2,\ldots,x_n$ in $\hat{X}\backslash\{z\}$ there do not exist neighborhoods U_1 of z and U_i of x_i, $i = 2,\ldots, n$, such that

$$\varepsilon D_G(y_1, y_2,\ldots, y_n) > 0, \qquad \varepsilon = \pm 1,$$

for all n-tuples

$$(y_1,\ldots, y_n) \in \prod_{i=1}^{n} U_i$$

for which $z \notin \{y_1,\ldots, y_n\}$. This implies the existence of sequences $\{z_k\}$, $\{\tilde{z}_k\}$, $\{x_{ik}\}$, $\{\tilde{x}_{ik}\} \subset \hat{X}\backslash\{z\}$ with $z_k \to z$, $\tilde{z}_k \to z$, $x_{ik} \to x_i$, $\tilde{x}_{ik} \to x_i$ for $k \to \infty$, $i = 2,\ldots, n$, such that

$$D_G(z_k, x_{2k},\ldots, x_{nk})\, D_G(\tilde{z}_k, \tilde{x}_{2k},\ldots, \tilde{x}_{nk}) < 0 \qquad \text{for all } k.$$

Then it follows from $z_k \neq z$, $\tilde{z}_k \neq z$ that

$$D_G(z_k, x_2,\ldots, x_n)\, D_G(\tilde{z}_k, x_2,\ldots, x_n) < 0 \qquad \text{for all } k.$$

Therefore the function $g_0 \in G$ defined by

$$g_0(x) := D_G(x, x_2,\ldots, x_n)$$

has a zero with a sign change at z and $n-1$ further zeros $x_2,\ldots, x_n$. Without loss of generality we may assume that there are sufficiently small positive constants δ_i such that $[x_i - \delta_i, x_i] \subset \hat{X}$. Then the Haar condition on $\hat{X}\backslash\{z\}$ implies the existence of a function $\tilde{g}_0 \in G$ with $\tilde{g}_0(x_i) = \operatorname{sgn} g_0(x_i - \delta_i)$. However, choosing a sufficiently small positive constant c, it turns out that $g_0 - c\tilde{g}_0$ has at least n zeros with a sign change in $\hat{X}$ which contradicts the hypotheses on G.

Second case. $Z = \{z\}$ and $z \in bd\hat{X}$. This case can be treated as the following case.

Third case. $Z = \{z_1, ..., z_m\}$, $m > 1$. Following the proof of statement (i) we have that $z_i \in bd\,\hat{X}$ for $i = 1, ..., m$. Now let n distinct points $x_1, ..., x_n \in \hat{X}$ be given. Without loss of generality we may assume that $\{x_1, ..., x_n\} \cap Z = \{x_1, ..., x_r\} = \{z_1, ..., z_r\}$. We furthermore may assume that for $i = 1, ..., r$ there are positive constants $\delta_i > 0$ with $[z_i, z_i + \delta_i) \subset \hat{X}$. Then $(z_i, z_i + \delta_i) \cap Z = \varnothing$ and $z_i \in bd\,\hat{X}$ implies that these sets are open neighborhoods of z_i. As in the first case we must only prove that for $i = 1, ..., r$ there are neighborhoods U_i of z_i such that

$$\varepsilon D_G(y_1, ..., y_r, x_{r+1}, ..., x_n) > 0, \qquad \varepsilon = \pm 1,$$

for all $n-$tuples $(y_1, ..., y_r, x_{r+1}, ..., x_n)$ with $y_i \in U_i \backslash \{z_i\}$, $i = 1, ..., r$. We define a function h in r variables by

$$h(y_1, ..., y_r) := D_G(y_1, ..., y_r, x_{r+1}, ..., x_n).$$

Then the statement is proved if for all $y_i \in (z_i, z_i + \delta_i)$ and $i = 1, ..., r$ the function h always assumes the same sign. Suppose that in each interval $(z_i, z_i + \delta_i)$ there are two points $\bar{y}_i$, $\bar{\bar{y}}_i, \bar{y}_i \leqslant \bar{\bar{y}}_i$ such that $h(\bar{y}_1, \bar{y}_2, ..., \bar{y}_r) \cdot h(\bar{\bar{y}}_1, \bar{\bar{y}}_2, ..., \bar{\bar{y}}_r) < 0$. Then h must have a zero at $(\tilde{y}_1, ..., \tilde{y}_r)$, where $\tilde{y}_i \in [\bar{y}_i, \bar{\bar{y}}_i]$, $i = 1, ..., r$. This means that $D_G(\tilde{y}_1, ..., \tilde{y}_r, x_{r+1}, ..., x_n) = 0$. But this is not possible because $(z_i, z_i + \delta_i) \cap Z = \varnothing$ implies that $\{\tilde{y}_1, ..., \tilde{y}_r, x_{r+1}, ..., x_n\} \cap Z = \varnothing$. This completes the proof.

It is not difficult to construct subspaces of $C(\hat{X})$ admitting pointwise-Lipschitz-continuous selections. This can be done as follows.

EXAMPLE 1. Let $G = \text{span}\{g_1, ..., g_n\}$ be a Chebyshev subspace of $C(\hat{X})$. Let $g_0 \in C(\hat{X})$, $g_0 \geqslant 0$, on $\hat{X}$ with exactly one zero on $\hat{X}$. Then the space $\tilde{G}$ defined by $\tilde{G} := \text{span}\{g_0 \cdot g_1, ..., g_0 \cdot g_n\}$ satisfies the hypotheses of Theorem 2.1.

In the case $\hat{X} = [a, b]$, a real compact interval Nürnberger [11] has shown that the weak Chebyshev property is necessary for existence of continuous selections. As has been defined before Lemma 2.2, an n-dimensional space G is said to be weak Chebyshev if each $g \in G$ has at most $n-1$ sign changes or, equivalently, if for a given basis $g_1, ..., g_n$ of G the inequality $\varepsilon D_G(x_1, ..., x_n) \geqslant 0$, $\varepsilon = \pm 1$, holds for all points $x_1 < \cdots < x_n$ in $\hat{X}$. In our general situation weak Chebyshev is no longer necessary for existence of continuous selections. Using Example 1 we can construct spaces G which are not Chebyshev and also not weak Chebyshev, however admit pointwise-Lipschitz-continuous selections for P_G. For $n = 1$ the existence of such spaces follows from results of Lazar *et al.* [8].

EXAMPLE 2. Let $\hat{X} = [-1, 1] \cup [2, 3] \cup [4, 5]$ and $G = \operatorname{span}\{g_1, ..., g_n\}$, where $g_i \in C(\hat{X})$ be defined by

$$g_i(x) := \begin{cases} x^{i-1} & \text{if } x \in [-1, 1] \cup [4, 5] \\ -x^{i-1} & \text{if } x \in [2, 3] \end{cases} \text{for } i = 1, ..., n.$$

Furthermore let $g_0 \in C(\hat{X})$ be defined by

$$g_0(x) := \begin{cases} |x| & \text{if } x \in [-1, 1] \\ 1 & \text{if } x \in [2, 3] \cup [4, 5]. \end{cases}$$

Then G is a Chebyshev subspace of $C(\hat{X})$. Using Example 1 we can show that the space $\tilde{G}$ defined by $\tilde{G} := \operatorname{span}\{g_0 \cdot g_1, ..., g_0 \cdot g_n\}$ admits a pointwise-Lipschitz-continuous selection for $P_{\tilde{G}}$. But $\tilde{G}$ is not Chebyshev and also not weak Chebyshev.

Finally we would like to ask if the complete converse to Lemma 1.1 will be true. In the case $\hat{X} = [a, b]$ the answer is "yes," because the following statements are equivalent as has been shown in [21]:

(i) Each non-zero $g \in G$ has at most n distinct zeros and at most $n-1$ zeros with a sign change in $[a, b]$.

(ii) G satisfies condition (1.2) and also the Haar condition on $[a, b] \setminus \{\tilde{x}\}$ for a certain $\tilde{x} \in [a, b]$.

Unfortunately the equivalence of these statements fails if $\hat{X}$ is an arbitrary compact subset of the real line as we have shown in [20] by an example. Therefore the statement of Theorem 2.1 does not yield the complete converse to Lemma 1.1. However, we conjecture that statement (ii) of Theorem 1.5 still holds if G satisfies the conditions (1.1) and (1.2) and each non-zero $g \in G$ has at most n distinct zeros on $\hat{X}$. Then using Lemma 2.2 and Theorem 1.7 we would obtain the converse to Lemma 1.1.

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